

Convexity Spaces. II. Separation

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This is the second of a series of three papers dealing with convexity spaces. In the first paper [1] we defined a convexity space and investigated some of its basic properties. Here we consider the separation and support of convex sets. Throughout the paper we will be dealing with a convexity space (X, \cdot) and the terminology and notation used will be those of [1]. In particular A^c denotes the complement of the set A in X and \setminus is used to denote set-theoretic difference.

1. CONVEX PAIRS

Our discussion of the separation of convex sets is based on the notion of a convex pair, which has its origin in the following result proved by Kakutani in [2]: Suppose that A and B are disjoint convex subsets of a real vector space. Then there exist disjoint convex sets C and D whose union is the whole space and such that $A \subset C$ and $B \subset D$.

(1) DEFINITION. A *convex pair* (C, D) is an unordered pair of complementary non-empty convex subsets of X , i.e., $C, D \subset X$ are non-empty and convex with $C \cap D = \emptyset$ and $C \cup D = X$.

(2) DEFINITION. The convex pair (C, D) *separates* the sets A and B if either $A \subset C$ and $B \subset D$ or $A \subset D$ and $B \subset C$.

(3) THEOREM. *Any two disjoint non-empty convex sets can be separated by a convex pair.*

Proof. Let A and B be disjoint non-empty convex sets, and denote by Ω the non-empty collection of all ordered pairs (A_i, B_i) , where A_i and B_i are disjoint convex sets with $A \subset A_i$ and $B \subset B_i$. Define a partial order \leq on Ω by writing $(A_i, B_i) \leq (A_j, B_j)$ whenever $A_i \subset A_j$ and $B_i \subset B_j$. If Σ is a non-empty chain in (Ω, \leq) , then Σ is bounded above by

$$\left(\bigcup_{(A_i, B_i) \in \Sigma} A_i, \bigcup_{(A_i, B_i) \in \Sigma} B_i \right).$$

Thus (Ω, \leq) is a non-empty inductive system and so by Zorn's Lemma possesses a maximal element (C, D) say. We show that (C, D) is a convex pair which separates A and B . To do this we need only show that $C \cup D = X$. Suppose that $C \cup D \neq X$ and let $x \notin C \cup D$. Then, by the maximality of (C, D) , it follows that

$$[C \cup x] = x \cup xC \cup C \approx D$$

and

$$[D \cup x] = x \cup xD \cup D \approx C.$$

Since $x \notin C \cup D$ and $C \cap D = \phi$ we must have $xC \approx D$ and $xD \approx C$, whence $D/C \approx C/D$ and $C = CC \approx DD = D$ which is impossible. Hence (C, D) is a convex pair which separates A and B .

(4) COROLLARY. *Convex pairs exist in any convexity space having more than one point.*

(5) THEOREM. *A necessary and sufficient condition for a set to be convex is that each point not belonging to the set can be separated from it by a convex pair.*

Proof. We prove only the sufficiency, the necessity following from (3). Suppose that the set A satisfies the condition of the theorem and is not convex. Then there exist points a, b, c with $c \in ab$ such that $a, b \in A$ and $c \notin A$. By hypothesis there exists a convex pair (C, D) with $A \subset C$ and $c \in D$. Then, since C is convex, $c \in ab \subset C$ which contradicts the fact that $C \cap D = \phi$.

(6) THEOREM. *Suppose that (C, D) is a convex pair. Then $\hat{C} \cap \hat{D}$ is linear.*

Proof. Suppose that $\hat{C} \cap \hat{D}$ is not linear. Then there exist $x, y \in \hat{C} \cap \hat{D}$ such that $x/y \notin \hat{C} \cap \hat{D}$; say $z \in x/y$ but $z \notin \hat{C} \cap \hat{D}$. It follows from (15) of [1] that $z \in \tilde{C} \cup \tilde{D}$. Thus, using (16) of [1], we deduce that

$$x \subset zy \subset (\tilde{C} \cup \tilde{D}) \cdot (\hat{C} \cap \hat{D}) \subset \tilde{C}\hat{C} \cup \tilde{D}\hat{D} = \tilde{C} \cup \tilde{D},$$

which contradicts $x \in \hat{C} \cap \hat{D}$.

(7) DEFINITION. The *associated flat* of the convex pair (C, D) is the set $\hat{C} \cap \hat{D}$.

It follows from (6) that associated flats are always linear. In section 3 of this paper we show that under quite general conditions the associated flat

$\hat{C} \cap \hat{D}$ of the convex pair (C, D) can be thought of as a hyperplane in X and the sets \hat{C} and \hat{D} as closed half-spaces. Without imposing some conditions it is possible for an associated flat to be either the empty set or the whole space as the following examples show.

Let X be the convexity space of real rationals with joins defined in the natural way and consider the convex pair (C, D) where $C = \{x \in X: x < \sqrt{2}\}$ and $D = \{x \in X: x > \sqrt{2}\}$. Then the associated flat $\hat{C} \cap \hat{D}$ is empty. In the next section we define complete convexity spaces where this situation cannot occur.

It is interesting to note that there exist convex pairs each of whose members is ubiquitous (i.e. they each have linear access equal to the whole space). It follows from (23) of [1] that if one member of a convex pair is ubiquitous then so too is the other one. Let X be the real vector space (with usual joins) of those real sequences, all but a finite number of whose terms are zero. Let C consist of those non-zero members of X whose last non-zero term is positive and let $D = C^c$. Then (C, D) is a convex pair and we show that D (and hence C) is ubiquitous. To see this let $c = a_1, \dots, a_n, 0, 0, \dots$ where $a_n > 0$ and let $d = a_1, \dots, a_n, -1, 0, 0, \dots$. Then c is a typical member of C and $cd \in D$ showing that $C \subset \hat{D}$ and $\hat{D} = X$. Thus the associated flat of (C, D) is the whole space. In paper III we show that this situation can only occur in infinite-dimensional convexity spaces.

(8) THEOREM. *Suppose that the convex pairs (C, D) and (E, F) have the same associated flat and that (C, D) has the property that whenever $c \in \tilde{C}$ and $d \in \tilde{D}$ then $cd \approx \hat{C} \cap \hat{D}$. Then either $\hat{C} = \hat{E}$ and $\hat{D} = \hat{F}$ or $\hat{C} = \hat{F}$ and $\hat{D} = \hat{E}$.*

Proof. If any of C, D, E, F is ubiquitous then the associated flat of both (C, D) and (E, F) is X whence $\hat{C} = \hat{D} = \hat{E} = \hat{F} = X$. Assume then that none of $\hat{C}, \hat{D}, \hat{E}, \hat{F}$ equals X and hence that none of $\tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}$ is empty. Note that

$$\tilde{C} \cup \tilde{D} = (\hat{C} \cap \hat{D})^c = (\hat{E} \cap \hat{F})^c = \tilde{E} \cup \tilde{F},$$

so assume without loss in generality that $\tilde{C} \approx \tilde{E}$, say $x \in \tilde{C} \cap \tilde{E}$. We will show that $\tilde{D} \subset \tilde{F}$ and $\tilde{C} \subset \tilde{E}$ which, together with the fact that \tilde{C} and \tilde{D} are disjoint, will show that $\hat{C} = \hat{E}$ and $\hat{D} = \hat{F}$. If $y \in \tilde{D}$, then by hypothesis $xy \approx \hat{C} \cap \hat{D} = \hat{E} \cap \hat{F} \subset \hat{E}^c$ and so $y \notin \hat{E}$. Thus $\tilde{D} \subset \tilde{F}$ and similarly $\tilde{C} \subset \tilde{E}$, whence $\hat{C} = \hat{E}$ and $\hat{D} = \hat{F}$. Hence by (19) of [1] we have

$$\hat{C} = \hat{\tilde{C}} = \hat{E} = \hat{E} \quad \text{and} \quad \hat{D} = \hat{\tilde{D}} = \hat{F} = \hat{F}$$

as required.

The theorem is not true if the condition on (C, D) is omitted, for in our

earlier example of the convexity space of real rationals the empty set is the associated flat of infinitely many convex pairs each giving rise to a different pair of linear accesses.

2. COMPLETENESS

(9) DEFINITION. The convexity space (X, \cdot) is *complete* if whenever A is convex, $a \in A$ and $b \notin A$, there exists $c \in [a, b]$ such that $ac \subset A$ and $bc \subset A^c$.

It is easily seen that every real vector space (when considered as a convexity space in the natural way) is complete in the sense of (9), but that a non-trivial vector space over the rational field is not.

(10) THEOREM. Suppose that A is a convex subset of a complete convexity space. Then either $A = \phi$, $A = X$ or $\hat{A} \setminus \tilde{A} \neq \phi$.

Proof. Suppose $A \neq \phi$, $A \neq X$ and that $a \in A$, $b \in A^c$. Then by completeness there exists $c \in [a, b]$ with $ac \subset A$ and $bc \subset A^c$. Thus

$$c \in \hat{A} \cap \hat{A}^c = \hat{A} \setminus \tilde{A} \quad \text{and} \quad \hat{A} \setminus \tilde{A} \neq \phi.$$

(11) COROLLARY. Each associated flat in a complete convexity space is non-empty.

Proof. Suppose (C, D) is a convex pair in a complete convexity space. Then $C \neq \phi$, $C \neq X$ and so $\hat{C} \setminus \tilde{C} = \hat{C} \cap \hat{D} \neq \phi$.

(12) LEMMA. Suppose A is convex and $a \in A$. Then $[a, b] \cap A^c$ is convex for any b .

Proof. If $b \in A$, then $[a, b] \cap A^c$ is empty and hence convex. Assume then that $b \in A^c$ and let $c, d \in [a, b] \cap A^c$. Suppose there exists $e \in cd \cap A$. Then $e \in [a, b]$ and

$$[a, b] = a \cup ae \cup e \cup eb \cup b \subset A \cup eb \cup b.$$

Since $c, d \in A^c$ it follows that $c, d \in eb \cup b$ and so

$$e \in cd \subset (eb \cup b)(eb \cup b) = eb \cup b$$

implying that $e = b$, and this is impossible because $e \in A$ and $b \in A^c$. Thus $cd \subset [a, b] \cap A^c$ and the convexity of $[a, b] \cap A^c$ is established.

(13) THEOREM. A necessary and sufficient condition for a convexity space to be complete is that whenever (C, D) is a convex pair with $c \in C$ and $d \in D$, then there exists $e \in [c, d]$ with $ec \subset C$ and $ed \subset D$.

Proof. We prove the sufficiency, the necessity following easily from (9). Suppose the convex pairs of a convexity space satisfy the condition of the theorem and that A is convex with $a \in A$ and $b \in A^c$. By the lemma $[a, b] \cap A^c$ is convex and thus $[a, b] \cap A$ and $[a, b] \cap A^c$ are disjoint non-empty convex sets. By (3) there exists a convex pair (C, D) such that $[a, b] \cap A \subset C$ and $[a, b] \cap A^c \subset D$ and by hypothesis there exists $c \in [a, b]$ such that $ea \in C$ and $eb \in D$. Thus $ea \in A$, for $ea \approx A^c$ would imply $ea \approx [a, b] \cap A^c \subset D$ which is impossible. Similarly $eb \in A^c$ and it follows that the space is complete.

(14) THEOREM. *Suppose that (C, D) and (E, F) are convex pairs in a complete space which have the same associated flat. Then either $\hat{C} = \hat{E}$ and $\hat{D} = \hat{F}$ or $\hat{C} = \hat{F}$ and $\hat{D} = \hat{E}$.*

Proof. If $c \in \tilde{C}$ and $d \in \tilde{D}$, then by completeness there exists $e \in [c, d]$ such that $ce \in C$ and $de \in D$. Hence $cd \approx e \in \hat{C} \cap \hat{D}$ and the result follows from (8).

3. HYPERPLANES

In this section we define a hyperplane in a convexity space and consider its relation with the concept of associated flat already discussed.

(15) DEFINITION. A *hyperplane* in (X, \cdot) is a non-empty maximal proper linear subset of X .

Zorn's Lemma together with the fact that singleton sets are linear shows that hyperplanes exist in convexity spaces with more than one point. It follows from (20) of [1] that if H is a hyperplane then $\hat{H} = H$ and $\tilde{H} = \phi$.

(16) LEMMA. *Suppose that M is non-empty and linear, and $x \notin M$ is such that $\{M \cup x\} = X$. Then M is a hyperplane.*

Proof. Let N be a linear set which properly contains M and let $y \in N \setminus M$. Then

$$y \in \{M \cup x\} = M \cup xM/M \cup M/x,$$

whence either $y \in xM/M$ or $y \in M/x$ and in either case $x \in \{M \cup y\}$. This shows that $N \supset \{M \cup y\} \supset \{M \cup x\} = X$, so $N = X$ and M is a hyperplane.

(17) THEOREM. *An associated flat of a convex pair in a complete convexity space is either a hyperplane or the whole space.*

Proof. Suppose that (C, D) is a convex pair in a complete space and that $\hat{C} \cap \hat{D} \neq X$. Then by (6) and (11) $\hat{C} \cap \hat{D}$ is a non-empty proper linear subset of X and there exists $x \in (\hat{C} \cap \hat{D})^c = \tilde{C} \cup \tilde{D}$, say $x \in \tilde{C}$. We show that $\{(\hat{C} \cap \hat{D}) \cup x\} = X$, whence by (16) $\hat{C} \cap \hat{D}$ is a hyperplane. Let $y \in \tilde{D}$. Then by completeness $xy \approx \hat{C} \cap \hat{D}$ and so $y \approx (\hat{C} \cap \hat{D})/x \in \{(\hat{C} \cap \hat{D}) \cup x\}$ which shows that $\tilde{D} \subset \{(\hat{C} \cap \hat{D}) \cup x\}$. Let $z \in \tilde{C}$. Then

$$(\hat{C} \cap \hat{D})/z \subset \tilde{D} \subset \{(\hat{C} \cap \hat{D}) \cup x\},$$

whence $z \approx \{(\hat{C} \cap \hat{D}) \cup x\}$ and $\tilde{C} \subset \{(\hat{C} \cap \hat{D}) \cup x\}$. We deduce that $\{(\hat{C} \cap \hat{D}) \cup x\} = X$ as required.

(18) THEOREM. *Every hyperplane is the associated flat of some convex pair.*

Proof. Let H be a hyperplane and let $x \notin H$. Then by (9) of [1] we have

$$X = \{x \cup H\} = xH/xH = H \cup H/x \cup xH/H.$$

Thus if $C = H \cup H/x$ and $D = xH/H$ then (C, D) is a convex pair and we prove that $\hat{D} = D \cup H$, $\tilde{D} = D$ which will show that $\hat{C} \cap \hat{D} = \tilde{D}^c \cap \hat{D} = H$. If $y \in H/x$ and $z \in D = xH/H$, then $H/y \approx zH/H$ and $yz \approx H$. Thus if $y \in H/x$ then $y \notin \hat{D}$ and since $HD \subset D$ it follows that $\hat{D} = D \cup H$. Also if $w \in yz \cap H$, then $wz \subset HD \subset D$ and $z \in \tilde{D}$, whence $D = \tilde{D}$. Thus $\hat{C} \cap \hat{D} = H$ as required.

With the above notation we see that if $y \in \tilde{C}$ and $z \in \tilde{D}$ then $yz \approx H$. It follows from (8) that if (E, F) is any convex pair whose associated flat is H , then either $\hat{C} = \hat{E}$ and $\hat{D} = \hat{F}$ or $\hat{C} = \hat{F}$ and $\hat{D} = \hat{E}$, i.e. the sets \hat{C} and \hat{D} are uniquely determined by H and do not depend upon the particular choice of x . We are thus led to the following definition.

(19) DEFINITION. Let H be a hyperplane which is the associated flat of a convex pair (C, D) . Then the sets \hat{C} and \hat{D} called *closed half-spaces* and more particularly the closed half-spaces *associated with* H .

It follows from the above remarks that each hyperplane determines two convex sets whose union is X and whose intersection is H . Also any segment joining a core point of one of these sets to a core point of the other set meets H . We conclude with a brief discussion of support hyperplanes in a convexity space.

(20) DEFINITION. The hyperplane H is said to be a *support hyperplane* to the set A at a if $a \in \hat{A} \cap H$ and A is contained in one of the closed half-spaces associated with H .

(21) THEOREM. *Suppose that A is a convex set in a complete convexity space and that $\tilde{A} \neq \phi$ and $a \in \hat{A} \setminus \tilde{A}$. Then there exists a support hyperplane to A at a .*

Proof. The sets \tilde{A} and a are disjoint non-empty convex sets and by (3) there exists a convex pair (C, D) with $\tilde{A} \subset C$ and $a \in D$. By (17) the associated flat $\hat{C} \cap \hat{D}$ of (C, D) is either a hyperplane or the whole space. However

$$\hat{D} = \hat{C}^c = \tilde{C}^c \subset (\tilde{A})^c = \tilde{A}^c \neq X$$

and so $\hat{C} \cap \hat{D}$ is a hyperplane. By (19) of [1]

$$A \subset \hat{A} = \hat{\tilde{A}} \subset \hat{C} \quad \text{and} \quad a \in \hat{A} \cap D \subset \hat{A} \cap (\hat{C} \cap \hat{D})$$

which shows that $\hat{C} \cap \hat{D}$ is a support hyperplane to A at a .

The condition $\tilde{A} \neq \phi$ cannot be omitted in the statement of the theorem. For example if A is a convex ubiquitous set for which $\tilde{A} = \phi$ and $a \in \hat{A}$, then there is no support hyperplane to A at a . However, if A is a non-empty convex set for which $\tilde{A} = \phi$ and $\{A\} \neq X$, then clearly there is a hyperplane which contains $\{A\}$ and this hyperplane is a support hyperplane to A at each point of \hat{A} .

REFERENCES

1. V. W. BRYANT AND R. J. WEBSTER, Convexity spaces I: the basic properties, *J. Math. Anal. and Appl.* **37** (1972), 206–213.
2. S. KAKUTANI, Ein Beweis des Satzes von M. Eidelheit über konvexe Menge, *Proc. Imp. Acad. (Tokyo)* **13** (1937), 93–94.